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A characterization of functional realizations of three-dimensional quantum groups

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Abstract. We study in an exhaustive and systematic way the functional realizations of quantum and classical algebras related with SU(2), SU(1,1) and the oscillator group Os(1). In order to achieve this purpose we have adopted a mapping that is a generalization of an algebra expansion procedure. We have not transformed the algebras themselves, but their representations. This method enables us not only to relate 'classical' algebras and their q-analogues, but also different-type algebras (whether they are quantum or not). We have obtained and classified all possible q-realizations, including as particular cases all those already known in the literature.

1. Introduction

Quantum groups have recently become an intense research field in both mathematics and physics [1–3]. There are several approachs to the study of deformations of universal enveloping algebras, as well as a widespread set of applications (an extensive review is given in [4] and references therein). We try in this work to get a deeper insight into the deforming maps point of view [5–7].

In this paper we centre our attention on the commutation relations of a quantum algebra. Our aim is to express the generators of that algebra as functions of the non-deformed Lie algebra generators. That is, we are embedding the quantum generators in the enveloping algebra of the 'classical' structure by means of an algebra expansion procedure. It is already known [8] that if we take the two-dimensional Euclidean group E(2) with generators $\{J_3, P_+, P_-\}$ and commutation relations

$$[J_3, P_\pm] = \pm P_\pm \qquad [P_+, P_-] = 0 \tag{1.1}$$

where $P_+P_- = P_x^2 + P_y^2 = p^2 = C$ is the Casimir element, we can reach the SU(2) algebra by defining SU(2) generators as an expansion of E(2) as follows:

$$J_3 = J_3 \qquad J_+ = P_+ f^+(J_3) \qquad J_- = f^-(J_3)P_- \,. \tag{1.2}$$

It is easy to verify that choosing $f^+(J_3) = f^-(J_3) = \mathrm{i}/(2p)(2J_3+1)$, we obtain $[J_3, J_\pm] = \pm J_\pm$ and $[J_+, J_-] = 2J_3$. In this paper we show that it is possible to develop a similar construction relating three-dimensional algebras and their q-deformed analogues. To get the explicit functional form of this quantum expansion we have to act on a representation space of the non-deformed algebra. Therefore, what we are actually doing is deforming the classical representations into the quantum

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ones. By choosing a certain mapping and imposing the commutation rules of the deformed algebra, we find and characterize all possible functional realizations of the quantum algebras under study.

In the following we carry out a systematic study of the quantum deformations of the algebras generating SU(2), SU(1, 1) and the oscillator group Os(1). These algebras appear if we consider three generators h, e, f such that $[h, e] = e$, $[h, f] = -f$. Including the identity (that always belongs to the enveloping algebra), the third non-trivial commutation relation (up to normalization) must be $[e, f] = \pm h$ or $[e, f] = I$ if it is required that they close a Lie algebra. Each of these possibilities gives rise, respectively, to the three afore-mentioned algebras, which have a precise physical meaning as underlying structures of kinematical geometries: SU(2) yields an elliptic geometry, SU(1, 1) a hyperbolic one and Os(1) is isomorphic to the extended Newton–Hooke group that expresses the symmetry properties of an oscillating universe [9]. The latter seems to play an outstanding role in what follows, perhaps due to the fact that Os(1) can be obtained from the other two by a contraction. On the other hand, it is well known that oscillator realizations can be used to build representations of other groups ('boson realizations') [10]. Here we shall always work with the 'n' representation, but the role of other Os(1) representations will be discussed elsewhere.

We deform these algebras in two directions: in section 2 we obtain the characterization of all possible q-analogues of each of these algebras. In this sense, an algebra and its q-deformed algebra will be said to be algebras of the same type (belonging to a class of q-algebras, in which by making $q = 1$ we recover the classical structure). Section 3 is devoted to analysing the properties of the so-called 'characteristic functionals'. These objects contain the conditions we have to impose on the deforming mappings to produce q-deformations. In section 4 we apply our method to deform one type of algebra into a different one. We thus obtain the characterization of all functional realizations among different-type algebras (quantum or classical). The results obtained allow us to build all quantum realizations. We recognize among them those already known in the literature. Besides this, all non-quantum realizations appear as $q = 1$ cases within this framework. Hence, this method provides a structured and rather flexible scheme to understand algebra deformations.

Throughout this paper, we will consider Jimbo deformations [11], and use the notation

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} \qquad \{x\}_q = \frac{q^x + q^{-x}}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}} .$$

With these definitions $[x]_q = [x]_{q^{-1}}$, and $\{x\}_q = \{x\}_{q^{-1}}$. Hereafter q will be a real parameter.

2. Functional realizations of $SU(2)_q$, $SU(1,1)_q$ and $Os(1)_q$

We shall introduce our method for $SU(2)_q$ with certain detail, while for the other two cases (i.e. $SU(1,1)_q$ and the oscilator group $Os(1)_q$, that can be dealt with along the same lines) we shall remark the differences and final results.

2.1. Realizations of $SU(2)_q$

Let us consider the Lie algebra of SU(2) given by

$$[J_3, J_\pm] = \pm J_\pm \tag{2.1a}$$
$$[J_+, J_-] = 2J_3 \tag{2.1b}$$

and its representations

$$J_3 \,|j,m\rangle = m\,|j,m\rangle \qquad J_\pm \,|j,m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\,|j,m \pm 1\rangle \tag{2.2}$$

where j is given by the eigenvalue of the Casimir operator

$$C = (J_3 - \tfrac{1}{2})^2 + J_+ J_- = (j + \tfrac{1}{2})^2 \tag{2.3}$$

and $-j \leqslant m \leqslant j$. We shall try to deform the commutation rules (2.1) into the quantum $SU(2)_q$ algebra [11]

$$[\mathcal{J}_3, \mathcal{J}_\pm] = \pm \mathcal{J}_\pm \tag{2.4a}$$
$$[\mathcal{J}_+, \mathcal{J}_-] = [2\mathcal{J}_3]_q \tag{2.4b}$$

by means of the operators J_3, $J_\pm$ defined in the representation space given by (2.2). That is, we deform the representation, so we have to keep in mind that the final expressions that we shall arrive at will be meaningful only when applied to vectors of such spaces. In the expansion process it is possible that even the dimension of the representation space could change. So this point of view (with some limitations) takes into account a wide range of possibilities, as we shall see.

Within the space (2.1), we have in general $\mathcal{J}_\pm = \mathcal{J}_\pm(J_\pm, J_3)$ and $\mathcal{J}_3 = \mathcal{J}_3(J_\pm, J_3)$, but here we shall adopt the simplest solution

$$\mathcal{J}_3 = J_3 \qquad \mathcal{J}_+ = J_+ f^+(J_3) \qquad \mathcal{J}_- = f^-(J_3) J_- \,. \tag{2.5}$$

In doing so we keep valid the commutator (2.1*a*), while f^+ and f^- are functions which remain to be found, with the condition that (2.4*b*) is to be verified. Observe that there also exists the possibility $\mathcal{J}_3 = J_3 + c$ without modifying (2.4*a*). But we are interested in the easiest realization, and when we choose $c = 0$ we simply avoid displacing the spectrum of J_3. We impose that on a generic ket $|j,m\rangle$ (2.4*b*) is satisfied:

$$[\mathcal{J}_+, \mathcal{J}_-]\,|jm\rangle = [2\mathcal{J}_3]_q\,|jm\rangle = [2J_3]_q\,|jm\rangle = [2m]_q\,|jm\rangle \,. \tag{2.6}$$

If we substitute $\mathcal{J}_\pm$ using (2.5) we get

$$(j+m)(j-m+1)f^+(m-1)f^-(m-1) - (j-m)(j+m+1)f^-(m)f^+(m) = [2m]_q \,. \tag{2.7}$$

Once the half-integer number j is fixed, this relationship is to be verified for every $-j < m \leqslant j$, while for $m = -j$ we have

$$f^-(-j)f^+(-j) = \frac{1}{2j}[2(-j)]_q \,. \tag{2.8}$$

In this way we have a recurrence given by (2.7) whose first term is (2.8). Therefore we get the solution

$$f^-(m)f^+(m) = -\frac{1}{(j-m)(j+m+1)}\{[2m]_q + [2(m-1)]_q + \cdots + [2(-j)]_q\}\,. \tag{2.9}$$

The sum

$$S = -\{[2m]_q + [2(m-1)]_q + \cdots + [2(-j)]_q\}$$

can be carried out easily by means of sums of geometric series and we get

$$S = \frac{q^{2\left(j+\frac{1}{2}\right)} + q^{-2\left(j+\frac{1}{2}\right)} - \left(q^{2\left(m+\frac{1}{2}\right)} + q^{-2\left(m+\frac{1}{2}\right)}\right)}{(q-q^{-1})^2} = [j-m]_q[j+m+1]_q\,.$$

Finally we have

$$f^-(m)f^+(m) = \frac{[j-m]_q[j+m+1]_q}{(j-m)\,(j+m+1)}\,. \tag{2.10}$$

We can change m for J_3, since they act in the same way on the ket $|j,m\rangle$. The operator defined by the diagonal elements (2.10) in the basis $|j,m\rangle$ will be called characteristic functional $\Lambda(j|j_q)$ and describes the deformations of the representation 'j' of SU(2) into that labelled 'j_q' of $\mathrm{SU}(2)_q$. Since this is the only condition on $f^\pm$, we have a great deal of freedom to choose f^+ and f^- separately. We show this as follows:

$$\begin{aligned} f^+_{a,b}(J_3) &= \left(\frac{[j-J_3]_q}{(j-J_3)}\right)^{a+\frac{1}{2}}\left(\frac{[j+J_3+1]_q}{(j+J_3+1)}\right)^{b+\frac{1}{2}} \\ f^-_{a,b}(J_3) &= \left(\frac{[j-J_3]_q}{(j-J_3)}\right)^{-a+\frac{1}{2}}\left(\frac{[j+J_3+1]_q}{(j+J_3+1)}\right)^{-b+\frac{1}{2}} \end{aligned} \qquad a,b\in\mathbb{R}\,. \tag{2.11}$$

These solutions include the ones given in [6]. Remark that in the limit $q\to 1$, $f^\pm \to 1$, as it should be. Now we take some 'natural' elections for $f^\pm$.

2.1.1. The standard realization ($a = b = 0$). If we impose that $\mathcal{J}_\pm$ be Hermitian conjugate, there exists, up to a sign, only one solution:

$$\begin{aligned} \mathcal{J}^a_3 &= J_3 \\ \mathcal{J}^a_+ &= J_+\sqrt{\frac{[j-J_3]_q[j+J_3+1]_q}{(j-J_3)\,(j+J_3+1)}} \\ \mathcal{J}^a_- &= \sqrt{\frac{[j-J_3]_q[j+J_3+1]_q}{(j-J_3)\,(j+J_3+1)}}\,J_-\,. \end{aligned} \tag{2.12}$$

We call this election the standard realization, and the action of the deformed operators is already well known [11]:

$$\mathcal{J}^a_3\,|j,m\rangle = m\,|j,m\rangle \qquad \mathcal{J}^a_\pm\,|j,m\rangle = \sqrt{[j\mp m]_q[j\pm m+1]_q}\,|j,m\pm 1\rangle\,. \tag{2.13}$$

2.1.2. Deforming only one generator ($a = b = -\frac{1}{2}$). Another interesting possibility is to keep the maximum of generators unchanged at the cost of missing the Hermiticity relation. If for example we choose $f^+(J_3) = 1$, we get rid of the square root:

$$\mathcal{J}_3^b = J_3 \qquad \mathcal{J}_+^b = J_+ \qquad \mathcal{J}_-^b = \frac{[j - J_3]_q [j + J_3 + 1]_q}{(j - J_3)\,(j + J_3 + 1)} J_- \,. \tag{2.14}$$

This realization is given by

$$\begin{aligned} \mathcal{J}_3^b \,|j,m\rangle &= m\,|j,m\rangle \\ \mathcal{J}_+^b \,|j,m\rangle &= \sqrt{(j-m)(j+m+1)}\,|j,m+1\rangle \\ \mathcal{J}_-^b \,|j,m\rangle &= \frac{[j+m]_q [j-m+1]_q}{\sqrt{(j+m)\,(j-m+1)}}\,|j,m-1\rangle \,. \end{aligned} \tag{2.15}$$

2.1.3. An asymmetric deformation ($a = \frac{1}{2}$, $b = -\frac{1}{2}$). Another 'natural' election is the following one:

$$\mathcal{J}_3^c = J_3 \qquad \mathcal{J}_+^c = J_+ \frac{[j - J_3]_q}{(j - J_3)} \qquad \mathcal{J}_-^c = \frac{[j + J_3 + 1]_q}{(j + J_3 + 1)} J_- \,. \tag{2.16}$$

This is just the representation given by Macfarlane [12] by means of differential operators in one angular variable

$$\mathcal{J}_+^c = \bar{e}^{\mathrm{i}\phi}[j + \mathrm{i}\partial_\phi]_q \qquad \mathcal{J}_-^c = \bar{e}^{-\mathrm{i}\phi}[j - \mathrm{i}\partial_\phi]_q \qquad \mathcal{J}_3^c = -\mathrm{i}\partial_\phi \tag{2.17}$$

and where the basis $|j,m\rangle$ is realized by the set of functions $\bar{e}^{\mathrm{i}m\phi}$.

2.1.4. The Casimir operator. The Casimir operator for $\mathrm{SU}(2)_q$ is [11]

$$C_q = [\mathcal{J}_3 - \tfrac{1}{2}]_q^{\;2} + \mathcal{J}_+ \mathcal{J}_- \,. \tag{2.18}$$

If we substitute the functional expression for the generators, we get

$$C_q = [J_3 - \tfrac{1}{2}]_q^{\;2} + J_+\, \Lambda(j|j_q)\, J_- \,. \tag{2.19}$$

Now we let this expression act on a ket $|j,m\rangle$ and the result is

$$C_q\,|j,m\rangle = [j + \tfrac{1}{2}]_q^2\,|j,m\rangle = [\sqrt{C}]_q^2 |j,m\rangle \,. \tag{2.20}$$

That is, the Casimir depends only on the functional $\Lambda(j|j_q)$, and its eigenvalues are just the q-analogues of the non-deformed operator ones.

Observe that the commutator $[\mathcal{J}_+, \mathcal{J}_-] = [2\mathcal{J}_3]_q$ appears as a difference $[\mathcal{J}_3 + \frac{1}{2}]_q^2 - [\mathcal{J}_3 - \frac{1}{2}]_q^2 = [2\mathcal{J}_3]_q$. We shall show in the following that these kinds of relations are a consequence of the quantum representation structure for the three types of algebras considered here. Let us compute $\|\mathcal{J}_-\,|j,m+1\rangle\,\|^2 - \|\mathcal{J}_-\,|j,m\rangle\,\|^2$ in the standard q-realization.

If we denote $\mathcal{J}_+ |j,m\rangle = \alpha_m |j,m+1\rangle$, where $\alpha_m > 0$, its Hermitic counterpart can be written as $\mathcal{J}_- |j,m+1\rangle = \alpha_m |j,m\rangle$. Making use of this notation

$$\begin{aligned}\langle j,m+1| \mathcal{J}_+\mathcal{J}_- |j,m+1\rangle &- \langle j,m| \mathcal{J}_+\mathcal{J}_- |j,m\rangle \\ &= \left(\langle j,m| \alpha_m^{-1}\mathcal{J}_-\right) \mathcal{J}_+ (\alpha_m |j,m\rangle) - \langle j,m| \mathcal{J}_+\mathcal{J}_- |j,m\rangle \\ &= \langle j,m| [\mathcal{J}_-, \mathcal{J}_+] |j,m\rangle = \langle j,m| - [2\mathcal{J}_3]_q |j,m\rangle . \end{aligned} \tag{2.21}$$

We may also compute this in a different way. Since

$$\mathcal{J}_+\mathcal{J}_- = -[\mathcal{J}_3 - \tfrac{1}{2}]_q^2 + [j + \tfrac{1}{2}]_q^2$$

we substitute and get

$$\begin{aligned}\langle j,m+1| \left(-[\mathcal{J}_3 - \tfrac{1}{2}]_q^2 + [j+\tfrac{1}{2}]_q^2\right) |j,m+1\rangle - \langle j,m| \left(-[\mathcal{J}_3 - \tfrac{1}{2}]_q^2\right. \\ \left. + [j+\tfrac{1}{2}]_q^2\right) |j,m\rangle = \langle j,m| \left(-[\mathcal{J}_3 + \tfrac{1}{2}]_q^2 + [\mathcal{J}_3 - \tfrac{1}{2}]_q^2\right) |j,m\rangle .\end{aligned}$$

These two results lead us to the identity

$$[\mathcal{J}_3 + \tfrac{1}{2}]_q^2 - [\mathcal{J}_3 - \tfrac{1}{2}]_q^2 = [2\mathcal{J}_3]_q . \tag{2.22}$$

We thus conclude that $[\mathcal{J}_+, \mathcal{J}_-] = [2\mathcal{J}_3]_q$ is an operator whose expected value on a certain state $|j,m\rangle$ expresses the difference of norms when $\mathcal{J}_-$ (or $\mathcal{J}_+$) acts on $|j,m+1\rangle$ and $|j,m\rangle$. This meaning remains valid for SU$(1,1)$ and Os(1)-type algebras.

2.2. *Realizations for the q-oscillator*

We follow the same pattern as in the previous case and begin with the commutation rules of the unidimensional oscilator algebra

$$[N, a^+] = a^+ \qquad [N, a] = -a \qquad [a, a^+] = I \tag{2.23}$$

and the representation space

$$N|n\rangle = n|n\rangle \qquad a^+|n\rangle = \sqrt{n+1}|n+1\rangle \qquad a|n\rangle = \sqrt{n}|n-1\rangle . \tag{2.24}$$

Here the Casimir operator is $C = N - a^+a$. Accordingly, the representation (2.24) corresponds to the eigenvalue $C = 0$. We shall call this the 'n' representation. The commutators for the q-deformed oscillator are [12, 13]

$$[\mathcal{N}, \tilde{a}^+] = \tilde{a}^+ \qquad Z[\mathcal{N}, \tilde{a}] = -\tilde{a} \tag{2.25a}$$

$$[\tilde{a}, \tilde{a}^+] = \frac{q^{N+\frac{1}{2}} + q^{-\left(N+\frac{1}{2}\right)}}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}} . \tag{2.25b}$$

We shall use the same ansatz to turn the representation (2.24) into the quantum-deformed (2.25):

$$\mathcal{N} = N \qquad \tilde{a}^+ = a^+ f^+(N) \qquad \tilde{a} = f^-(N)\, a . \tag{2.26}$$

Note that in (2.26) the number operator has been chosen such that the lowest eigenvalue has not been deformed and that we have started with commutation rules (2.25) symmetric under the replacement $q \to q^{-1}$ (compare in this respect with that given in [14]). If we apply (2.25*b*) on a state $|n\rangle$, $n \neq 0$ we obtain

$$(n+1)f^-(n)f^+(n) - nf^+(n-1)f^-(n-1) = \{n+\tfrac{1}{2}\}_q \qquad (2.27)$$

and for $n=0$ this is $f^-(0)f^+(0) = \{\tfrac{1}{2}\}_q$. As an immediate consequence we have

$$f^-(n)f^+(n) = \frac{1}{(n+1)}\left\{\{n+\tfrac{1}{2}\}_q + \{(n-1)+\tfrac{1}{2}\}_q + \ldots + \{\tfrac{1}{2}\}_q\right\}. \qquad (2.28)$$

This sum can be computed easily and we obtain

$$f^-(n)f^+(n) = \frac{[n+1]_q}{(n+1)}. \qquad (2.29)$$

The operator defined by the diagonal elements (2.29) is the characteristic functional $\Lambda(n|n_q)$, and it is the relevant object in the deforming process. The general solutions of (2.29) can be displayed as

$$f^-(N) = \left(\frac{[N+1]_q}{(N+1)}\right)^{\frac{1}{2}-\lambda} \qquad f^+(N) = \left(\frac{[N+1]_q}{(N+1)}\right)^{\frac{1}{2}+\lambda} \qquad \lambda \in \mathbb{R}. \qquad (2.30)$$

2.2.1. Some explicit realizations. In particular if $\tilde{a}^+$ and $\tilde{a}$ are Hermitian-conjugated we obtain the standard realization ($\lambda = 0$) [12]

$$\begin{aligned} \mathcal{N}_0|n\rangle &= n|n\rangle \\ \tilde{a}_0^+|n\rangle &= a^+\sqrt{\frac{[N+1]_q}{(N+1)}}|n\rangle = \sqrt{[n+1]_q}\,|n+1\rangle \\ \tilde{a}_0|n\rangle &= \sqrt{\frac{[N+1]_q}{(N+1)}}\,a|n\rangle = \sqrt{[n]_q}\,|n-1\rangle\,. \end{aligned} \qquad (2.31)$$

Another case of interest consists in deforming only one operator, for example $\tilde{a}^+$ ($\lambda = \tfrac{1}{2}$). In this case we have

$$\begin{aligned} \mathcal{N}_{1/2}|n\rangle &= n|n\rangle \\ \tilde{a}_{1/2}^+|n\rangle &= a^+\frac{[N+1]_q}{(N+1)}|n\rangle = \frac{[n+1]_q}{\sqrt{n+1}}|n+1\rangle \\ \tilde{a}_{1/2}|n\rangle &= a|n\rangle = \sqrt{n}\,|n-1\rangle\,. \end{aligned} \qquad (2.32)$$

2.2.2. The Casimir operator. The q-Casimir of the quantum algebra $\mathrm{Os}(1)_q$ is $C_q = [N]_q - \tilde{a}^+\tilde{a}$. Its eigenvalue for the representation (2.32) is $C_q = 0$. Remark that the commutator $[\tilde{a}, \tilde{a}^+]$ in (2.25*b*) can be expressed as the difference of the action of the q-number operator $[N]_q$ on two sucessive states

$$[N+1]_q - [N]_q = \{N+\tfrac{1}{2}\}_q\,.$$

2.3. Realizations for $SU(1,1)_q$

The commutation rules for $SU(1,1)_q$ are

$$[K_3, K_\pm] = \pm K_\pm \qquad [K_+, K_-] = -[2K_3]_q\,. \tag{2.33}$$

Here we consider the positive discrete series representations characterized by the eigenvalue $l > 0$ of the Casimir operator

$$C = (K_3 - \tfrac{1}{2})^2 - K_+K_- = (l - \tfrac{1}{2})^2\,.$$

(Negative discrete series representations $l < 0$ can be given an analogous treatment.) Representations corresponding to these series take the form

$$\begin{aligned} K_3|l, z\rangle &= z|l, z\rangle \\ K_+|l, z\rangle &= \sqrt{(z + l)(z - l + 1)}|l, z\rangle \\ K_-|l, z\rangle &= \sqrt{(z - l)(z + l - 1)}|l, z\rangle \end{aligned} \tag{2.34}$$

where $z = l, l + 1, l + 2, \ldots$.

The quantum deformation of this algebra has the following commutation rules [15]:

$$[\mathcal{K}_3, \mathcal{K}_\pm] = \pm\mathcal{K}_\pm \qquad [\mathcal{K}_+, \mathcal{K}_-] = -[2\mathcal{K}_3]_q\,. \tag{2.35}$$

As always, we shall deform the representation (2.33) according to the ansatz

$$\mathcal{K}_3 = K_3 \qquad \mathcal{K}_+ = K_+f^+(K_3) \qquad \mathcal{K}_- = f^-(K_3)K_-\,. \tag{2.36}$$

We get the same kind of solution for the product f^+f^- of the form

$$f^+(z)f^-(z) = \frac{1}{(z + k)(z - k + 1)}\Big\{-[2(z - l - k)]_q - [2(z - l - k - 1)]_q - \ldots - [2(-l)]_q\Big\} \tag{2.37}$$

from which we get the final expression

$$\Lambda(l|l_q) = f^+(K_3)f^-(K_3) = \frac{[K_3 - l + 1]_q[K_3 + l]_q}{(K_3 - l + 1)(K_3 + l)}\,. \tag{2.38}$$

For the standard realization we have the symmetric solution $f^+(K_3) = f^-(K_3)$ corresponding to Hermitian-conjugated operators $\mathcal{K}_+$ and $\mathcal{K}_-$. In this case the actions of the generators $\mathcal{K}_3$, $\mathcal{K}_\pm$ on the basis kets are given by the q-analogues of (2.34). The Casimir operator for the quantum group is $C_q = [\mathcal{K}_3 - \frac{1}{2}]_q^2 - \mathcal{K}_+\mathcal{K}_- = [l - \frac{1}{2}]_q^2$ [15].

We remark that we have taken non-trivial realizations on the discrete series characterized by $l > 0$. However, there exists the limit of the characteristic functional when $l \to 0$

$$\lim_{l\to 0}\Lambda(l|l_q) = \frac{[K_3 + 1]_q[K_3]_q}{(K_3 + 1)K_3}\,. \tag{2.39}$$

This is just the functional associated to the $l = 0$ continuous series representations of $SU(1,1)_q$ obtained in [6] and [16].

Note that in this case a similar relationship to that given for $SU(2)_q$ holds, namely

$$[\mathcal{K}_3 + \tfrac{1}{2}]_q^2 - [\mathcal{K}_3 - \tfrac{1}{2}]_q^2 = [2\mathcal{K}_3]_q\,. \tag{2.40}$$

3. Properties of the characteristic functional

In principle we can apply the aforementioned method to deform a (classical or quantum) algebra into another (classical or quantum) algebra of the same or different type. The deforming process will be characterized by a functional $\Lambda(a|b)$, where a is for the initial representation and b for the final one; if any of these algebras is quantum it will be explicitly designed, for example, by b_q. In this section we show some simple properties of the functionals $\Lambda(a|b)$ which will be helpful in obtaining and characterizing in a systematic way all deformations between any pair of algebras.

Property 1. All the realizations obtained for different choices of the deforming functions $g^{\pm}$ corresponding to the same functional, $\Lambda(a|b)$, are equivalent.

We can show this property for the three cases $\mathrm{SU}(2)_q$, $\mathrm{SU}(1,1)_q$ and $\mathrm{Os}(1)_q$ at the same time, since their (classical) structure is similar: it is made up of two step generators, e for the upper and f for the lower, together with a diagonal one, h. The deformed generators $\tilde{e}, \tilde{f}, \tilde{h}$, take the form

$$\tilde{h} = h + \lambda \qquad \tilde{e} = eg^{+}(h) \qquad \tilde{f} = g^{-}(h)f\,. \tag{3.1}$$

The standard realization $\tilde{e}_s, \tilde{f}_s, \tilde{h}_s$ is characterized by $g^{+}(h) = g^{-}(h)$ or by the equivalent relation $\tilde{e}_s^{+} = \tilde{f}_s$. First we shall show the equivalence of any realization $\{\tilde{e}, \tilde{f}\}$ with the standard $\{\tilde{e}_s, \tilde{f}_s\}$ through a Hermitic operator $K(h)$ that depends only on h:

$$K^{-1}(h)\tilde{e}\ K(h) = \tilde{e}_s \qquad K^{-1}(h)\tilde{f}\ K(h) = \tilde{f}_s\,. \tag{3.2}$$

Since $\tilde{e}_s^{+} = \tilde{f}_s$,

$$(K^{-1}(h)\tilde{f}K(h))^{\dagger} = K^{-1}(h)\tilde{e}K(h) = K(h)\tilde{f}^{\dagger}K^{-1}(h)\,. \tag{3.3}$$

From (3.2) and (3.3) we get

$$K(h)^2 eg^{-}(h) = eg^{+}(h)K(h)^2\,. \tag{3.4}$$

If we design by $|h_i\rangle$ a discrete basis, where $h|h_i\rangle = h_i|h_i\rangle$ and $|h_0\rangle$ is the lowest state, we have that, within this space, relation (3.4) gives rise to the following recurrence:

$$K(h+1)^2 = \frac{g^{+}(h)}{g^{-}(h)}K(h)^2\,. \tag{3.5}$$

Let us employ the notation $\langle h_i|K(h)|h_i\rangle = K(h_i)$. Then, given an arbitrary value $K(h_0)$, we obtain the solution

$$K(h_i) = \sqrt{\frac{g^{+}(h_i-1)g^{+}(h_i-2)\ldots g^{+}(h_0+1)}{g^{-}(h_i-1)g^{-}(h_i-2)\ldots g^{-}(h_0+1)}}K(h_0)\,. \tag{3.6}$$

This operator gives us the equivalence between any deformed realization $g^{\pm}$ and the standard $g_s^{\pm}$; hence by transitivity we have really shown the equivalence between any pair of realizations.

Property 2. If there exists $\Lambda(a|b)$ and $\Lambda(b|c)$ then $\Lambda(a|c)$ also exists and is given by

$$\Lambda(a|c) = \Lambda(a|b)\,\Lambda(b|c)\,. \tag{3.7}$$

This transitive property is a direct consequence of the fact that any functional Λ is defined by a product g^+g^- which obviously shares this property.

Property 3. If $\Lambda(a|b)$ has an inverse, then

$$\Lambda(b|a) = \Lambda(a|b)^{-1}\,. \tag{3.8}$$

Property 3 is a straightforward corollary of property 2.

Property 4. If some (or both) $a_{(q)}$, $b_{(q)}$ are quantum algebras, then

$$\lim_{q\to 1}\Lambda(a_{(q)}|b_{(q)}) = \Lambda(a|b)\,. \tag{3.9}$$

It is easy to show this property as a consequence of the way that the functionals have been constructed. Property 4 can be useful in deriving some classical relationships starting with more general deformed expressions, as we shall see in the following sections.

4. Functional realizations relating different algebras

We may also look for the functional realizations relating algebras of different type by applying the same expansion method. We assume that diagonal operators of the two algebras are related by a linear transformation, and the constant is chosen to ensure that the vector with eigenvalue 0 under the lowering operator coincides in both representations. This constant will label the thus-obtained realizations.

4.1. Oscillator realizations of SU(1,1)$_q$

If we define

$$\mathcal{K}_3 = N + l \qquad \mathcal{K}_+ = a^+g^+(N) \qquad \mathcal{K}_- = g^-(N)a \tag{4.1}$$

with $l > 0$, we have an action on a number state space such that $\mathcal{K}_3\,|0\rangle = l\,|0\rangle$, and $\mathcal{K}_-\,|0\rangle = 0$. A straightforward calculation shows that $[\mathcal{K}_3, \mathcal{K}_\pm] = \pm\mathcal{K}_\pm$ and (2.33) leads us to the finite-difference equation given by

$$ng^+(n-1)g^-(n-1) - (n+1)g^-(n)g^+(n) = -[2(n+l)]_q\,. \tag{4.2}$$

The solution is easily found by the recursion method used before and the expression for the characteristic functional is

$$\Lambda(n|l_q) = g^+(N)g^-(N) = \frac{1}{(N+1)}[N+2l]_q[N+1]_q\,. \tag{4.3}$$

An explicit action on the number state space describes the q-analogue of the positive discrete series of SU(1,1). For instance, by choosing $g^+(N) = g^-(N)$, we obtain

$$\begin{aligned}
\mathcal{K}_3\,|n\rangle &= (n+l)\,|n\rangle \\
\mathcal{K}_+\,|n\rangle &= \sqrt{[n+1]_q[2l+n]_q}\,|n+1\rangle \\
\mathcal{K}_-\,|n\rangle &= \sqrt{[n]_q[2l+n-1]_q}\,|n-1\rangle\,.
\end{aligned} \tag{4.4}$$

So we can identify $|n\rangle = |l, z = n+l\rangle$, according to (2.34).

4.2. Oscillator realizations of $SU(2)_q$

We may apply the same scheme to the $SU(2)_q$ algebra by writing

$$\mathcal{J}_3 = N - j \qquad j = 0, \tfrac{1}{2}, 1 \ldots \tag{4.5a}$$

$$\mathcal{J}_+ = a^+ g^+(N) \tag{4.5b}$$

$$\mathcal{J}_- = g^-(N) a . \tag{4.5c}$$

(The condition $\mathcal{J}_3 |0\rangle = -j |0\rangle$ is satisfied.) The finite-difference equation is then

$$n g^+(n-1) g^-(n-1) - (n+1) g^-(n) g^+(n) = [2(n-j)]_q \tag{4.6}$$

and any solution must fulfil the following condition:

$$\Lambda(n|j_q) = g^+(N)\, g^-(N) = \frac{1}{(N+1)} [2j - N]_q [N+1]_q . \tag{4.7}$$

The symmetric solution yields an action on the states $|n\rangle$ described by the equations

$$\begin{aligned} \mathcal{J}_3 |n\rangle &= (n-j)\, |n\rangle \\ \mathcal{J}_+ |n\rangle &= \sqrt{[n+1]_q [2j-n]_q}\, |n+1\rangle \\ \mathcal{J}_- |n\rangle &= \sqrt{[n]_q [2j-n+1]_q}\, |n-1\rangle . \end{aligned} \tag{4.8}$$

If we consider only the subspace spanned by the number states $|n\rangle$, $(n = 0, 1, \ldots, 2j)$ labelled as vectors $|j, m = n - j\rangle$ we immediately deduce that

$$\begin{aligned} \mathcal{J}_3 |j,m\rangle &= m\, |j,m\rangle \\ \mathcal{J}_+ |j,m\rangle &= \sqrt{[j+m+1]_q [j-m]_q}\, |j,m+1\rangle \\ \mathcal{J}_- |j,m\rangle &= \sqrt{[j+m]_q [j-m+1]_q}\, |j,m-1\rangle . \end{aligned} \tag{4.9}$$

These equations are formally identical to the standard realization of $SU(2)_q$.

4.3. $SU(2)_q$ realizations on $SU(1,1)$

If we take $SU(1,1)$ as the starting group, we build the functional realizations of $SU(2)_q$ on the positive discrete series labelled by l and supported by vectors $|l, z\rangle$ defining

$$\mathcal{J}_3 = K_3 - j - l \quad j = 0, \tfrac{1}{2}, 1 \ldots \qquad \mathcal{J}_+ = K^+ g^+(K_3) \qquad \mathcal{J}_- = g^-(K_3)\, K_- . \tag{4.10}$$

We thus obtain the following equation:

$$(z-l)(z+l-1) g^+(z-1) g^-(z-1) - (z+l)(z-l+1) g^-(z) g^+(z) = [2(z-j-l)]_q \tag{4.11}$$

which is equivalent to writing

$$\Lambda(l|j_q) = g^+(K_3) g^-(K_3) = \frac{[K_3 - l + 1]_q [2j + l - K_3]_q}{(K_3 - l + 1)(K_3 + l)} . \tag{4.12}$$

The Hermitic solution acts on $|l,z\rangle$ as follows:

$$\begin{aligned}\mathcal{J}_3\,|l,z\rangle &= (z-j-l)\,|l,z\rangle \\ \mathcal{J}_+\,|l,z\rangle &= \sqrt{[z-l+1]_q[2j+l-z]_q}\,|l,z+1\rangle \\ \mathcal{J}_-\,|l,z\rangle &= \sqrt{[z-l]_q[2j+l-z+1]_q}\,|l,z-1\rangle\,.\end{aligned} \tag{4.13}$$

The first of these equations expresses the equivalence $|l,z\rangle \equiv |j,m\rangle$, where $m = z-l-j$. If we replace z in (4.13) we obtain again the standard action (2.13).

4.4. Q-oscillator realizations on SU(1,1)

In this case we define the relation between $\mathcal{N}$ and K_3 as the reciprocal relation to the first of the (4.1) equations:

$$\mathcal{N} = K_3 - l \qquad \tilde{a}^+ = K^+ g^+(N) \qquad \tilde{a} = g^-(N)\,K_-\,. \tag{4.14}$$

Therefore, we preserve $\mathcal{N}\,|l\rangle = 0\,|l\rangle$.

The characterization of the functional realizations is given by

$$\Lambda(l|n_q) = g^+(K_3)\,g^-(K_3) = \frac{[K_3-l+1]_q}{(K_3-l+1)(K_3+l)}\,. \tag{4.15}$$

The action of the standard election on the discrete series basis vectors emerges as

$$\begin{aligned}\mathcal{N}_0\,|z,l\rangle &= (z-l)\,|z,l\rangle \\ \tilde{a}_0^+\,|z,l\rangle &= \sqrt{[z-l+1]_q}\,|z+1,l\rangle \\ \tilde{a}_0\,|z,l\rangle &= \sqrt{[z-l]_q}\,|z-1,l\rangle\end{aligned} \tag{4.16}$$

and with the labelling $|n\rangle \equiv |z,l\rangle$ with $n = z-l$ we obtain (2.31).

We want to stress here in which cases the existence of the inverse for the functional realization is guaranteed. It is easy to check that all the q-analogues of these three algebras built on their own representation space (section 2) are invertible. However, the results in the present section show that there exist non-invertible functional realizations. The problems arise with the compact group, SU(2) and its q-analogue. The latter can be constructed from SU(1,1) and also from the oscillator group, but both functionals (4.7) and (4.12) are not invertible since their action on certain states gives a zero eigenvalue ($\Lambda(n|j_q)\,|n=2j\rangle = 0$ and $\Lambda(l|j_q)\,|l,z=2j+l\rangle = 0$). This is an expected result: we may reduce an infinite representation space belonging to a non-compact group by choosing a subspace of states (the lowest ones in our case). On this 'compact' subspace we may build a realization of the compact group. The reversal is not possible, and hence the realization must be non-invertible.

4.5. Realizations involving two quantum algebras

It is straightforward to characterize the functional realizations of a quantum algebra in terms of another quantum algebra belonging to a different class by taking into account the properties we have just shown in section 3.

4.5.1. $SU(1,1)_q$ realizations on $Os(1)_q$. The deformation described in section 4.1 which relates

$$\mathrm{Os}(1) \stackrel{\Lambda(n|l_q)}{\longrightarrow} \mathrm{SU}(1,1)_q$$

with characteristic function given by (4.3) can be seen as a composition of deformations:

$$\mathrm{Os}(1) \stackrel{\Lambda(n|n_q)}{\longrightarrow} \mathrm{Os}(1)_q \stackrel{\Lambda(n_q|l_q)}{\longrightarrow} \mathrm{SU}(1,1)_q\,.$$

Since we have shown that $\Lambda(n|l_q) = \Lambda(n|n_q)\Lambda(n_q|l_q)$ (equation (3.7)) under the appropriate translation of diagonal operators, we find that the deformation

$$\mathcal{K}_3 = \mathcal{N} + l = N + l \qquad \mathcal{K}_+ = \tilde{a}^+ g^+(\mathcal{N}) \qquad \mathcal{K}_- = g^-(\mathcal{N})\tilde{a} \tag{4.17}$$

is characterized by

$$\Lambda(n_q|l_q) = g^+(\mathcal{N})g^-(\mathcal{N}) = [\mathcal{N} + 2l]_q\,. \tag{4.18}$$

The most common known realizations between these two quantum algebras are included in (4.18). Imposing the Hermitic condition we obtain all the q-analogues of the Holstein–Primakoff realizations of SU(1,1) [17–19]. We also obtain Dyson-type realizations choosing $g^+(\mathcal{N}) = I$ or $g^-(\mathcal{N}) = I$. For instance, in the first case, we may write as a realization of $\mathrm{SU}(1,1)_q$

$$\mathcal{K}_3^d = \mathcal{N} + l = N + l \qquad \mathcal{K}_+^d = \tilde{a}^+ \qquad \mathcal{K}_-^d = [N + 2l]_q\tilde{a}\,. \tag{4.19}$$

It is important to remark that transformations (4.18) turn the $\mathrm{Os}(1)_q$ q-Casimir into that of $\mathrm{SU}(1,1)_q$. If we start with

$$[\mathcal{N}]_q - \tilde{a}^+\tilde{a} = 0 \tag{4.20}$$

and substitute these deformed generators in terms of $\mathcal{K}_3, \mathcal{K}_\pm$ (equation (4.17)), we obtain

$$[\mathcal{K}_3 - l]_q - \frac{1}{[\mathcal{K}_3 + l - 1]_q}\mathcal{K}_+\mathcal{K}_- = 0$$

or

$$[\mathcal{K}_3 + l - 1]_q[\mathcal{K}_3 - l]_q - \mathcal{K}_+\mathcal{K}_- = 0\,.$$

Taking into account the fact that $\mathcal{K}_3$ is diagonal and making use of the following identity for q-numbers:

$$[a - b]_q[a + b - 1]_q = [a - \tfrac{1}{2}]_q^2 - [b - \tfrac{1}{2}]_q^2 \tag{4.21}$$

we get

$$[\mathcal{K}_3 - \tfrac{1}{2}]_q^2 - \mathcal{K}_+\mathcal{K}_- = [l - \tfrac{1}{2}]_q^2$$

which is the q-Casimir for $\mathrm{SU}(1,1)_q$ with the correct l. Analogous inverse substitutions can be used in expansions studied in section 2 to derive the q-Casimir operators from the classical invariants. Once again, the characteristic functional is the object that contains all the deformation properties.

4.5.2. Q-oscillator realizations on $SU(1,1)_q$. For any $|n\rangle$ and l, the eigenvalue of $\Lambda(n_q|l_q)|n\rangle$ is positive. Therefore we can rewrite (4.17) expressing the inverse deformation as

$$\begin{aligned}\mathcal{N} &= \mathcal{K}_3 - l \\ \tilde{a}^+ &= \mathcal{K}_+ f^+(\mathcal{K}_3) = \mathcal{K}_+ g^+(\mathcal{K}_3 - l)^{-1} \\ \tilde{a} &= f^-(\mathcal{K}_3)\mathcal{K}_- = g^-(\mathcal{K}_3 - l)^{-1}\mathcal{K}_- .\end{aligned} \tag{4.22}$$

The characteristic functional will be

$$\Lambda(l_q|n_q) = \Lambda(n_q|l_q)^{-1} = \frac{1}{[\mathcal{K}_3 + l]_q} . \tag{4.23}$$

4.5.3. $SU(2)_q$ functional realizations on $Os(1)_q$. Let us start now with the chain

$$\mathrm{Os}(1) \xrightarrow{\Lambda(n|n_q)} \mathrm{Os}(1)_q \xrightarrow{\Lambda(n_q|j_q)} \mathrm{SU}(2)_q$$

where $\Lambda(n_q|j_q)$ is the unknown functional. We define

$$\mathcal{J}_3 = \mathcal{N} - j \qquad j = 0, \tfrac{1}{2}, 1 \ldots \qquad \mathcal{J}_+ = \tilde{a}^+ g^+(\mathcal{N}) \qquad \mathcal{J}_- = g^-(\mathcal{N})\tilde{a} . \tag{4.24}$$

From the straightforward transformation $\mathrm{Os}(1) \xrightarrow{\Lambda(n|j_q)} \mathrm{SU}(2)_q$ characterized by (4.7), we reach the expression for $\Lambda(n_q|j_q)$

$$\Lambda(n_q|j_q) = g^+(\mathcal{N})g^-(\mathcal{N}) = [2j - N]_q . \tag{4.25}$$

This equation again contains all the known single q-boson realizations of $\mathrm{SU}(2)_q$. Holstein–Primakoff realizations suggested in [19] and [20] are obtained by identifying $g^+(\mathcal{N}) = g^-(\mathcal{N}) = \sqrt{[2j - N]_q}$. Dyson realizations obtained by using a coherent-state technique in [20], are contained in (4.25) by making $g^-(\mathcal{N}) = I$, and the existence of an intertwining operator relating them with Holstein–Primakoff realizations is a particular case of the general equivalence property.

Observe also that if we substitute (4.24) into the $\mathrm{Os}(1)_q$ Casimir $[\mathcal{N}]_q - \tilde{a}^+\tilde{a} = 0$ and let it act on SU(2) representation space we get

$$[\mathcal{J}_3 + j]_q [j - \mathcal{J}_3 - 1]_q - \mathcal{J}_+\mathcal{J}_- = 0 .$$

But from (4.21)

$$[\mathcal{J}_3 + j]_q [j - \mathcal{J}_3 - 1]_q = [j + \tfrac{1}{2}]_q^2 - [\mathcal{J}_3 - \tfrac{1}{2}]_q^2$$

and we finally obtain

$$[\mathcal{J}_3 - \tfrac{1}{2}]_q^2 + \mathcal{J}_+\mathcal{J}_- = [j + \tfrac{1}{2}]_q^2$$

which is the q-Casimir for $\mathrm{SU}(2)_q$ (equation (2.18)).

4.5.4. $SU(2)_q$ functional realizations on $SU(1,1)_q$ representation space. The only remaining expansion is the $SU(1,1)_q \xrightarrow{\Lambda(l_q|j_q)} SU(2)_q$ transformation. By applying the known characteristic functionals $\Lambda(l|l_q)$ and $\Lambda(l|j_q)$ from (2.38) and (4.12), respectively, and regarding the deformation definitions as

$$\begin{aligned} \mathcal{J}_3 &= \mathcal{K}_3 - j - l = \mathcal{K}_3 - j - l \qquad j = 0, \tfrac{1}{2}, 1 \ldots \\ \mathcal{J}_+ &= \mathcal{K}^+ g^+(\mathcal{K}_3) \qquad \mathcal{J}_- = g^-(\mathcal{K}_3)\, \mathcal{K}_- \end{aligned} \tag{4.26}$$

we finally obtain

$$\Lambda(l_q|j_q) = g^+(\mathcal{K}_3) g^-(\mathcal{K}_3) = \frac{[2j + l - \mathcal{K}_3]_q}{[\mathcal{K}_3 + l]_q}\,. \tag{4.27}$$

The Hermitic choice guides us to the following realization:

$$\mathcal{J}_3 = \mathcal{K}_3 - j - l \qquad \mathcal{J}_+ = \mathcal{K}^+ \sqrt{\frac{[2j + l - \mathcal{K}_3]_q}{[\mathcal{K}_3 + l]_q}} \qquad \mathcal{J}_- = \sqrt{\frac{[2j + l - \mathcal{K}_3]_q}{[\mathcal{K}_3 + l]_q}} \mathcal{K}_-\,. \tag{4.28}$$

It is important to emphasize the fact that this is not the only way to reach (4.27). For instance, we might have chosen the path

$$SU(1,1)_q \xrightarrow{\Lambda(l_q|n_q)} Os(1)_q \xrightarrow{\Lambda(n_q|j_q)} SU(2)_q\,.$$

Casimirs can also be related. If we now start with

$$[\mathcal{K}_3 - \tfrac{1}{2}]_q^2 - \mathcal{K}_+\mathcal{K}_- = [l - \tfrac{1}{2}]_q^2$$

and substitute $\mathcal{K}_3, \mathcal{K}_\pm$ using (4.26) and (4.27), we have

$$[\mathcal{J}_3 + j + l - \tfrac{1}{2}]_q^2 [j - \mathcal{J}_3 + 1]_q - [\mathcal{J}_3 + j + 2l - 1]_q \mathcal{J}_+\mathcal{J}_- = [-\mathcal{J}_3 + j + 1]_q [l - \tfrac{1}{2}]_q^2\,.$$

After a lengthy but straightforward calculation on the space $|j, m\rangle$ this equality leads us again to the right q-Casimir $[\mathcal{J}_3 - \frac{1}{2}]_q^2 + \mathcal{J}_+\mathcal{J}_- = [j + \frac{1}{2}]_q^2$.

4.6. Functional realizations relating classical algebras

Property 4 in section 2 implies that if we compute the limit $q \to 1$ of a certain characteristic functional, the thus-obtained result characterizes all the possible classical functional realizations between the two involved non-deformed algebras. This limit turns out to be the identity if we are working with same type algebras. Hence we obtain, for instance, the non-deformed realizations of SU(2) in terms of the oscillator algebra by computing the classical limit of (4.7):

$$\Lambda(n|j) = \lim_{q \to 1} \Lambda(n|j_q) = g_1^+(N) g_1^-(N) = 2j - N\,. \tag{4.29}$$

The analogue of (4.5) is now

$$J_3 = N - j \qquad j = 0, \tfrac{1}{2}, 1 \ldots \qquad J_+ = a^+ g_0^+(N) \qquad J_- = g_0^-(N)\, a\,. \tag{4.30}$$

The Hermitic solution gives us the classical Holstein–Primakoff realizations of SU(2) [17], and the so-called Dyson representation can be reached by choosing one of the g_0 functions to be equal to I. Besides, we have obtained infinitely more realizations. None of them is invertible, due to the same difficulty that appeared when we studied $\mathrm{SU}(2)_q$.

We want to point out that this result is again unique whatever the path to deform an oscillator-type algebra (quantum or not) into $\mathrm{SU}(2)_q$. If we look at (4.5*c*), we see that the characteristic functional of this deformation can be written as $\Lambda(n|j_q) = \Lambda(n|n_q)\Lambda(n_q|j_q)$, and $\lim_{q\to 1}\Lambda(n|n_q) = 1$ (same type structures). Then

$$\lim_{q\to 1}\Lambda(n|j_q) = \lim_{q\to 1}\Lambda(n_q|j_q) = 2j - N\,. \tag{4.31}$$

In the following we summarize the classical characterizations among the rest of the types of algebras analysed here.

4.6.1. SU(1,1) realizations on the oscillator algebra. Defining the classical analogue to (4.1) as

$$K_3 = N + l \qquad K_+ = a^+ g_0^+(N) \qquad K_- = g_0^-(N)a \tag{4.32}$$

expression (4.7) leads us to

$$\Lambda(n|l) = g_0^+(N)g_0^-(N) = N + 2l\,. \tag{4.33}$$

Holstein–Primakoff and Dyson realizations are obtained as in the just-studied SU(2) case.

4.6.2. SU(2) realizations on the SU(1,1) positive discrete series. We now begin from (4.10) and write

$$\begin{aligned} &J_3 = K_3 - j - l \qquad j = 0, \tfrac{1}{2}, 1 \ldots \\ &J_+ = K^+ g_0^+(K_3) \qquad J_- = g_0^-(K_3)\ K_-\,. \end{aligned} \tag{4.34}$$

The corresponding limit is

$$\Lambda(l|j) = g_0^+(K_3)g_0^-(K_3) = \frac{(2j + l - K_3)}{(K_3 + l)}\,. \tag{4.35}$$

We may now write a classical action of SU(2) (with the Hermitic constrain) on discrete series vectors $|l, z\rangle$:

$$\begin{aligned} J_3\,|l,z\rangle &= (z - j - l)\,|l,z\rangle \\ J_+\,|l,z\rangle &= \sqrt{(z - l + 1)(2j + l - z)}\,|l, z+1\rangle \\ J_-\,|l,z\rangle &= \sqrt{(z - l)(2j + l - z + 1)}\,|l, z-1\rangle\,. \end{aligned} \tag{4.36}$$

Transformations given by (4.31), (4.33) and (4.35) allow us to transform one given Casimir into another following the reasoning outlined in section 4.5.

5. Final remarks

Figure 1 shows the whole expansion scheme. Lower and upper triangles represents, respectively, classical and deformed structures. Arrows are pointing towards the algebra we obtain after each expansion. Invertible mappings associate SU(1, 1), the oscillator group and their q-analogues, and are drawn with full lines. The rest of realizations (all ending in SU(2) structures and denoted with broken lines) are not invertible. Corresponding characteristic functionals are written and classical realizations are obtained by projecting on the lower plane ($q = 1$).

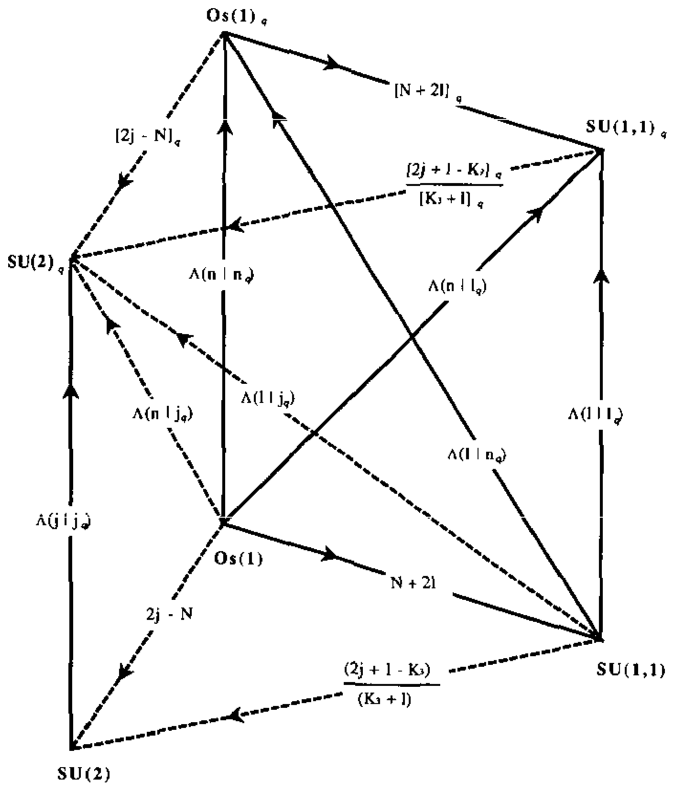

Figure 1. A triangular prism containing all expansions that relate SU(2), SU(1, 1) and Os(1)-type algebras. Quantum ($q \neq 1$) structures lies on the upper face, while $q = 1$ ('classical') cases correspond to the basis. Characteristic functionals labelling all possible q-realizations are explicitly given.

There are some aspects that remain to be studied to complete the expansion framework outlined here. First of all, to investigate Hopf structures underlying these realizations and to study the q root of unity problems, since we believe that the latter could be a way to relate, on a deeper level, these three algebras. We have also used throughout this paper the 'n' representation of the oscillator group. However, other non-equivalent representations could play a relevant role in the study of more general links among these three types of algebras. Remark also that all new realizations described here can be used to generalize quantum coherent-state theory, following the method given in [19].

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References

[1] Sklyanin E, Takhtajan L and Fadeev L 1979 *Theor. Math. Phys.* **40** 194
[2] Kulish P and Reshetikhin N 1983 *J. Sov. Math.* **23** 2435
[3] Drinfeld V G 1986 Quantum groups *Proc. Int. Congr. of Mathematics (MRSI Berkeley)* p 798
[4] Biedenharn L C 1991 An overview of quantum groups *Proce. XVIII Int. Colloquium of Group Theoretical Methods in Physics (Lecture Notes in Physics 382)* (Berlin: Springer)
[5] Curtright T L 1990 *Physics and Geometry* ed L-L Chau and W Nahm (New York: Plenum)
[6] Curtright T L and Zachos C K 1990 *Phys. Lett.* **243B** 237
[7] Fairlie D B 1990 *J. Phys. A: Math. Gen.* **23** L183
[8] Gilmore H 1974 *Lie Groups, Lie Algebras and Some of Their Applications* (New York: Wiley)
[9] Bacry H and Lévy-Leblond J M 1968 *J. Math. Phys.* **9** 1605
[10] Sun C P 1987 *J. Phys. A: Math. Gen.* **20** 4551
[11] Jimbo M 1985 *Lett. Math. Phys.* **10** 63; 1986 *Lett. Math. Phys.* **11** 247
[12] Macfarlane A J 1989 *J. Phys. A: Math. Gen.* **22** 4581
[13] Biedenharn L C 1989 *J. Phys. A: Math. Gen.* **22** L873
[14] Filippov A T, Gangopadhay D and Isaev A.P 1991 *J. Phys. A: Math. Gen.* **24** L63
[15] Kulish P P and Damashinski E V 1990 *J. Phys. A: Math. Gen.* **23** L415
[16] Bernard D and Clair A Le 1989 *Phys. Lett.* **227B** 417
[17] Holstein T and Primakoff H 1940 *Phys. Rev.* **58** 1048
[18] Chaichian M, Ellinas D and Kulish P P 1990 *Phys. Rev. Lett.* **65** 980
[19] Katriel J andSolomon A I 1991 *J. Phys. A: Math. Gen.* **24** 2093
[20] Yu Zurong 1991 *J. Phys. A: Math. Gen.* **24** L1321